ON AN APPROXIMATE METHOD OF ANALYSIS OF CIRCULAR CYLINDRICAL SHELLS

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In the case of absence of surface loads the analysis of circular cylindrical shells can be reduced [1] to the problem of solving the equation

$$(1+4a^2)\frac{\partial^8\Phi}{\partial\xi^8}+4(1+a^2)\frac{\partial^8\Phi}{\partial\xi^6\,\partial\theta^2}+[6+a^2(1-\sigma^2)]\frac{\partial^8\Phi}{\partial\xi^4\,\partial\theta^4}+4\frac{\partial^8\Phi}{\partial\xi^2\,\partial\theta^6}+\frac{\partial^8\Phi}{\partial\theta^8}+$$
$$+(8-2\sigma^2)\frac{\partial^6\Phi}{\partial\xi^4\,\partial\theta^2}+8\frac{\partial^6\Phi}{\partial\xi^2\,\partial\theta^4}+2\frac{\partial^6\Phi}{\partial\theta^6}+(1-\sigma^2)\left(\frac{1}{a^2}+4\right)\frac{\partial^4\Phi}{\partial\xi^4}+4\frac{\partial^4\Phi}{\partial\xi^2\,\partial\theta^2}+\frac{\partial^4\Phi}{\partial\theta^4}=0$$

where $a^2 = h^2/3r^2$ ($2h$ is the thickness of the shell, r is the radius of the shell), while Φ is a potential function.

The displacements of the middle surface of the shell can be expressed in terms of the function Φ by means of the formulas

$$u=\left\{\frac{\partial^3}{\partial\xi\,\partial\theta^2}+\sigma\frac{\partial^3}{\partial\xi^3}+a^2\left[\frac{(1+\sigma)(2-\sigma)}{1-\sigma}\frac{\partial^5}{\partial\xi^3\,\partial\theta^2}+\frac{1+\sigma}{1-\sigma}\frac{\partial^5}{\partial\xi\,\partial\theta^4}+4\sigma\frac{\partial^3}{\partial\xi^3}+\frac{2\sigma}{1-\sigma}\frac{\partial^3}{\partial\xi\,\partial\theta^2}\right]\right\}\Phi$$

$$v=\left\{(2+\sigma)\frac{\partial^3}{\partial\xi^2\,\partial\theta}+\frac{\partial^3}{\partial\theta^3}-a^2\left[\frac{2(2-\sigma)}{1-\sigma}\frac{\partial^5}{\partial\xi^4\,\partial\theta}+\frac{4-3\sigma+\sigma^2}{1-\sigma}\frac{\partial^5}{\partial\xi^2\,\partial\theta^3}+\frac{\partial^5}{\partial\theta^5}\right]\right\}\Phi \qquad (0.1)$$

$$w=\left\{\frac{\partial^4}{\partial\xi^4}+2\frac{\partial^4}{\partial\xi^2\,\partial\theta^2}+\frac{\partial^4}{\partial\theta^4}+a^2\left[\frac{\partial^4}{\partial\xi^4}+\frac{2(2-2\sigma+\sigma^2)}{1-\sigma}\frac{\partial^4}{\partial\xi^2\,\partial\theta^2}+\frac{\partial^4}{\partial\theta^4}\right]\right\}\Phi$$

Substituting u, v, w into the geometric relations

$$\varepsilon_1=\frac{1}{r}\frac{\partial u}{\partial\xi},\qquad \varepsilon_2=\frac{1}{r}\left(\frac{\partial v}{\partial\theta}-w\right),\qquad \omega=\frac{1}{r}\left(\frac{\partial v}{\partial\xi}+\frac{\partial u}{\partial\theta}\right)$$
$$\varkappa_1=\frac{1}{r^2}\frac{\partial^2 w}{\partial\xi^2},\qquad \varkappa_2=\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta}+v\right),\qquad \tau=\frac{1}{r^2}\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\theta}+v\right)$$

and then into the elasticity relations

$$T_1=\frac{2Eh}{1-\sigma^2}(\varepsilon_1+\sigma\varepsilon_2),\ T_2=\frac{2Eh}{1-\sigma^2}(\varepsilon_2+\sigma\varepsilon_1),\quad S_1=\frac{2Eh}{1+\sigma}\left(\frac{\omega}{2}+\frac{h^2}{3}\frac{\tau}{r}\right),\ S_2=-\frac{2Eh}{1+\sigma}\frac{\omega}{2}$$

$$G_1=-\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_1+\sigma\varkappa_2),\quad G_2=-\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_2+\sigma\varkappa_1),\quad H_1=-H_2=\frac{2Eh^3}{3(1+\sigma)}\tau$$

we can express the stress resultants T_1, T_2, S_1, S_2 and the stress couples G_1, G_2, H_1, H_2 in terms of the potential function Φ. From the equilibrium conditions

$$\frac{\partial H_1}{\partial\xi}-\frac{\partial G_2}{\partial\theta}+rN_2=0,\qquad \frac{\partial H_2}{\partial\theta}+\frac{\partial G_1}{\partial\xi}-rN_1=0$$

we can find also the shear forces N_1 and N_2.

Consider a closed circular cylindrical shell. Expanding the potential function Φ into a trigonometric series in terms of θ

$$\Phi=\sum_{m=0}^{\infty}[\varphi_m'(\xi)\cos m\theta+\varphi_m''(\xi)\sin m\theta] \tag{0.2}$$

we obtain for the determination of ϕ_m' and ϕ_m'' the same differential equation (omitting the primes and the subscript m)

$$(1+4a^2)\frac{d^8\varphi}{d\xi^8}-4(1+a^2)m^2\frac{d^6\varphi}{d\xi^6}+\left[6m^4+a^2(1-\sigma^2)m^4-(8-2\sigma^2)m^2+\right.$$
$$\left.+(1-\sigma^2)\left(\frac{1}{a^2}+4\right)\right]\frac{d^4\varphi}{d\xi^4}-4m^2(m^2-1)^2\frac{d^2\varphi}{d\xi^2}+m^4(m^2-1)^2\varphi=0 \tag{0.3}$$

Integrating this equation we obtain the mth term of the expansion (0.2), each item of which corresponds to a certain state of stress and strain in the circular cylindrical shell. Using for any m the integrals ϕ_m' and ϕ_m'', we obtain the expressions for the displacements, rotation angles, stress resultants and stress couples in the form of trigonometric series. Substituting

$$m=a^{-\mu} \tag{0.4}$$

into Equation (0.3) and seeking to obtain its solution in the form

$$\varphi=Ae^{k\xi}\qquad (A=\text{const}) \tag{0.5}$$

we obtain the characteristic equation

$$(1+4a^2)k^8-4(1+a^2)a^{-2\mu}k^6+\left\{[6+a^2(1-\sigma^2)]a^{-4\mu}-(8-2\sigma^2)a^{-2\mu}+\right.$$
$$\left.+(1-\sigma^2)\left(\frac{1}{a^2}+4\right)\right\}k^4-4a^{-2\mu}(a^{-2\mu}-1)^2k^2+a^{-4\mu}(a^{-2\mu}-1)^2=0 \tag{0.6}$$

Since the solution of Equation (0.6) and the formulas (0.1) are very cumbersome, the question arises as to approximate methods of analysing cylindrical shells and of estimating the inaccuracies involved in the use of approximate formulas. The asymptotic estimation of the roots of the characteristic equation is the basis for the entire theory of circular cylindrical shells as developed in the monograph of Gol'denveizer referred to above. The author has given only the first approximation using, as stated by himself, non-rigorous methods. The purpose of the present paper is to

1. improve the accuracy and the basis of the first approximation;
2. indicate a method for deriving successive approximations;
3. develop an effective method for estimation of the asymptotic ($a \to 0$) inaccuracy corresponding to a given order of approximation.

These objectives will be achieved by an asymptotic analysis of the roots of algebraic equations with coefficients depending on a small parameter a.

1. On a method of asymptotic solution of some algebraic equations.

1. Consider an equation of the nth degree with respect to k:

$$f(a, k) \equiv b_0 k^n + b_1 k^{n-1} + \ldots + b_r k^{n-r} + \ldots + b_{n-1} k + b_n = 0 \qquad (1.1)$$

with coefficients b_r depending on a parameter a in a manner expressed by the formula

$$b_r = A_r a^{\alpha_r} + A_r' a^{\alpha_r'} + \ldots = a_r(a)\, a^{\alpha_r} \qquad (r = 0, 1, \ldots, n) \qquad (1.2)$$

where all A_r, A_r', ... are independent of a, $A_r \neq 0$ and $\alpha_r < \alpha_r' < \ldots$, so that $\lim a_r(a) = A_r \neq 0$ when $a \to 0$.

Definition. If a quantity $B(a)$ can be represented in the form $B(a) = b(a) a^m$ and $\lim b(a) = B^\circ \neq 0$, when $a \to 0$, then we shall call m the order of B.

Thus α_r represents the order of the coefficient b_r of Equation (1.1).

We start from the hypothetical form

$$k = \varkappa(a)\, a^s \qquad (\lim \varkappa(a) = K \neq 0 \text{ when } a \to 0)$$

for a root of Equation (1.1).

Since, at a sufficiently small value of a, the limit value K differs but little from $\kappa(a)$, the quantity Ka^s can be considered to represent, in first approximation, a root of Equation (1.1).

We concentrate first on the problem of determining s, the order of the root, and K.

2. To find a root of the equation in first approximation, we substitute into (1.1), $\kappa(a)a^s$ for k and the expressions (1.2) for the coefficients b_r. Multiplying then the resulting equation by a^t, where t is a number to be chosen later, we obtain

$$a_0 a^{\alpha_0+ns+t}\kappa^n + a_1 a^{\alpha_1+(n-1)s+t}\kappa^{n-1} + \cdots + a_r a^{\alpha_r+(n-r)s+t}\kappa^{n-r} + \cdots a_n a^{\alpha_n+t} = 0 \tag{1.3}$$

Since the roots of any algebraic equation are continuous functions of its coefficients, $K = \lim \kappa(a)$ at $a \to 0$ represents a root of the limiting (at $a \to 0$) equation, provided, of course, that these roots exist and differ from zero.

We will select s and t in such a manner that the corresponding limiting equation should exist, without degenerating into an identity, and have non-vanishing roots, i.e. that in the coefficients of Equation (1.3) at least two degree exponents be equal to zero, while all other exponents are positive numbers. Substituting one of such pairs of values of s, t into (1.3), passing to the limit as $a \to 0$ and dividing by K^{n-m}, we obtain an equation for K:

$$A_l K^{m-l} + A_{l'} K^{m-l'} + \cdots + A_m = 0. \tag{1.4}$$

Thus the problem of finding a root in first approximation reduces to that of finding a pair of numbers s, t and to the solution of the limiting equation (1.4) of degree $m - l$, which is, in general, a simpler one than the original equation.

3. For the determination of s the following procedure can be used. With the numbers α_0, α_1, ..., α_r, ..., α_n, defined by the relations (1.2), we construct the following table:

$$\begin{array}{cccccc}
\frac{\alpha_0-\alpha_1}{1} & \frac{\alpha_0-\alpha_2}{2} & \cdots & \frac{\alpha_0-\alpha_{m'}}{m'} & \cdots & \frac{\alpha_0-\alpha_n}{n} \\
 & \frac{\alpha_1-\alpha_2}{1} & \cdots & \frac{\alpha_1-\alpha_{m'}}{m'-1} & \cdots & \frac{\alpha_1-\alpha_n}{n-1} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & \cdots & \frac{\alpha_m-\alpha_{m'}}{m'-m} & \cdots & \frac{\alpha_m-\alpha_n}{n-m} \\
 & & & \cdots & \cdots & \cdots \\
 & & & & & \frac{\alpha_{n-1}-\alpha_n}{1}
\end{array} \tag{1.5}$$

a) In each line of this table we choose the largest number $(\alpha_p - \alpha_{p'})/(p' - p)$, and if it appears several times in that line, we choose among them the one which corresponds to the largest subscript p';

b) In each column we choose the smallest number $(a_q - a_{q'})/(q' - q)$, and if it appears several times in that column, we choose the one which corresponds to the smallest subscript q.

It can be shown that the table (1.5) possesses the following properties (which are being stated here without proof):

1) Among the chosen numbers there will be such, which have to be selected both by virtue of the condition (a) and by virtue of the condition (b). We shall call such numbers nodal numbers of the table.

2) If for reasons of convenience we count the lines of the table 0 to $n - 1$ from the top to the bottom, and the columns 1 to n from the left to the right, then a nodal number will appear in the zero line, and the same is true of the column n (it can happen that these two numbers are actually one and the same element of the table; in such a case this number is the only nodal number of the table).

3) If the number $(a_p - a_{p'})/(p' - p)$, corresponding to the index p', is the nodal number of the pth line, then there will be no nodal numbers in the $(p + 1)$th, $(p + 2)$th, ..., $(p' - 1)$th lines, but the p'th line will have a nodal number.

Note. It is thus not necessary to write down completely the entire table (1.5) in order to find the nodal numbers. It is sufficient to write down the line carrying the number 0 and to choose in it, according to condition (a), the largest number $(a_0 - a_l)/l$ corresponding to a certain subscript l. This will be the first nodal number U_1. Then we have to write down the lth line and to treat it in the same way, which yields the second nodal number U_2, and so forth.

4) The nodal numbers thus obtained are decreasing:

$$U_1 > U_2 > \dots$$

5) The desired values s_1, s_2, ... of the quantity s are expressed in terms of the nodal numbers of the table (1.5) in the following manner:

$$s_1 = -U_1, \qquad s_2 = -U_2, \dots$$

4. The value t_i, corresponding to any value s_i, of the quantity t has to be chosen in such a manner as to render equal to zero the smallest degree exponent in the coefficients of Equation (1.3); there will be at least two such exponents, which is ascertained by a corresponding choice of the s_i values.

Since Ka^s is a root of Equation (1.1) in first approximation and since it differs but a little from the exact root, if only a is sufficiently small, the quantity a^s characterizes the rate of decrease (increase) of the root as $a \to 0$. It is not difficult to show that the degree of the limiting equation (1.4), obtained for a certain s_i, equals $q - p$, the denominator of the nodal number $(a_p - a_q)/(q - p)$, corresponding to s_i, so that the sum of the degrees of all limiting equations equals n. Thus all roots of Equation (1.1) can be subdivided, with respect to their asymptotic (as $a \to 0$) properties, into as many groups as there are nodal numbers in the table (1.5).

5. Assume that Ka^s represents the first approximation of a root of Equation (1.1), so that $k \approx Ka^s$. To find the second approximation, we substitute $k = Ka^s + z$ into (1.1), and of all approximate values of z, obtained by the same method, we take the value $Z = K'a^{s'}$ with the power exponent $s' > s$. It is possible to prove the existence of such a z.

For example, the lowest term

$$Z_M = K'a^{s'_M}$$

(term with the largest exponent) possesses this property [2].

For the second approximation of the root we have

$$k \approx Ka^s + K'a^{s'}$$

Continuing this procedure we find that any root of Equation (1.1) can be written in the form of a series

$$k = Ka^s + K'a^{s'} + K''a^{s''} + \dots \tag{1.6}$$

where $s < s' < s'' \dots$ (which is sufficient for asymptotic convergence).

Note. The problem which we have stated can be generalized to include the cases when $a \to a_0 \neq 0$ or $a \to \infty$; these problems can be reduced to the one considered here by introducing a new parameter

$$a' = a - a_0 \qquad \text{or} \qquad a'' = 1/a$$

6. The question which we have raised here has been apparently originally discussed by Newton, who has given a geometric method of solving the problem. The description of this method, the theorem proving the convergence of the procedure and some applications can be found in Chebotarev's paper [2]. Bugaev formulated this solution in analytical terms [3]. The work of Newton and that of Bugaev has not become generally known, however. In particular, the author of the present paper

became aware of them when his own work on the subject was fundamentally completed. It should be noted that the method developed above and based upon the use of the table (1.5) and its properties, achieves the purpose more speedily than Bugaev's method.

2. Estimation of the asymptotic error. 1. Taking the series (1.6) as a root k_1 and retaining in it the first two terms, we may write

$$k_i \approx K_i a^s \left(1 + \frac{K_i'}{K_i} a^{s'-s}\right)$$

We shall call the quantity $\epsilon(a) = a^{s'-s}$ the asymptotic error of the first approximation of the root considered.

It is often possible to avoid the necessity of establishing the numbers s' in the procedure of determining the asymptotic error. Let us substitute, into the left-hand member of (1.1), first approximation of the root $K_i a^s$, as well as the quantity Ka^s, considering K as some parameter. Let

$$f(a, K_i a^s) \equiv a^{q_l} F(a), \qquad f(a, Ka^s) \equiv a^q \Phi(a)$$

where

$$\lim F(a) = F^\circ \neq 0, \qquad \lim \Phi(a) = \Phi^\circ \neq 0 \qquad \text{as} \quad a \to 0$$

There is, in a number of cases, a simple relationship between q_l, q and $s' - s$, and then the determination of s' becomes unnecessary. Let us study this question.

2. Suppose we write $f(a, Ka^s)$ in the form

$$f(a, Ka^s) \equiv \varphi(K) a^q + \varphi_1(K) a^{q_1} + \varphi_2(K) a^{q_2} + \ldots \quad (q < q_1 < q_2 < \ldots) \tag{2.1}$$

Since $K_i a^s$ represents first approximation of the root k_i considered, K_i satisfies the equation $\phi(K) = 0$, so that we have $\phi(K_i) = 0$. If $\phi_1(K_i) = \phi_2(K_i) = \ldots = \phi_{l-1}(K_i) = 0$, while $\phi_l(K_i) \neq 0$, then the order of the expression $f(a, K_i a^s)$ equals q_l (assuming that $K_i a^s$ is not an exact root of the original equation, so that l actually exists).

We substitute also $K_i a^s + z$ into Equation (1.1):

$$f(a, K_i a^s + z) = \frac{1}{n!} \frac{\partial^n f(a, K_i a^s)}{\partial K^n} a^{-ns} z^n + \frac{1}{(n-1)!} \frac{\partial^{n-1} f(a, K_i a^s)}{\partial K^{n-1}} a^{-(n-1)s} z^{n-1} + \\ + \cdots + \frac{1}{t!} \frac{\partial^t f(a, K_i a^s)}{\partial K^t} a^{-ts} z^t + \cdots + \frac{1}{1!} \frac{\partial f(a, K_i a^s)}{\partial K} a^{-s} z + f(a, K_i a^s) = 0 \tag{2.2}$$

Suppose β_{n-t} is the order of the coefficient of z^t, while r is the multiplicity of the root K_i in the equation $\phi(K) = 0$. Differentiating the relation (2.1) with respect to K r times, we find for the coefficient of z^r in Equation (2.2)

$$\frac{1}{r!}\frac{\partial^r f(a, K_i a^s)}{\partial K^r} a^{-rs} \equiv \frac{1}{r!}[\varphi^{(r)}(K_i) a^q + \varphi_1^{(r)}(K_i) a^{q_1} + \cdots] a^{-rs}$$

Since $\phi^{(r)}(K_i) \neq 0$, the order of the coefficient of z^r will be $\beta_{n-r} = q - rs$, while, as stated already above, the order of the free term will be $\beta_n = q_l (l \geqslant 1)$. In the procedure of determining from Equation (2.2) the second term Z of the series (1.6) for k_i, it is necessary, as in Section 1, to construct a table of the type (1.5), replacing Z for the time being by $Z_M = K' a^{s_M'}$ with the largest exponent s_M. Its nodal number will be

$$U_M' = -s_M' = \min\left(\frac{\beta_{n-1} - \beta_n}{1}, \frac{\beta_{n-2} - \beta_n}{2}, \ldots, \frac{\beta_{n-r} - \beta_n}{r}, \ldots, \frac{\beta_0 - \beta_n}{n}\right) \tag{2.3}$$

It is possible to show [4] that for all $u > r$

$$\frac{\beta_{n-u} - \beta_n}{u} > \frac{\beta_{n-r} - \beta_n}{r} \tag{2.4}$$

Therefore the degree of the equation, from which K_i' is to be determined, does not exceed r, the multiplicity of the root K_i, and

$$-s_M' \leqslant \frac{\beta_{n-r} - \beta_n}{r}$$

But

$$\frac{\beta_{n-r} - \beta_n}{r} = \frac{q - rs - q_l}{r} = \frac{q - q_l}{r} - s.$$

so that

$$s_M' \geqslant \frac{q_l - q}{r} + s$$

Therefore the order of the asymptotic error $S_M' - s$ will not be smaller than $(q_l - q)/r$, or

$$\varepsilon(a) = a^{s'-s} = a^{s_M'-s} \leqslant a^{(q_l - q)/r}$$

It must be noted, however, that in some cases the use of the lowest term Z_M only does not permit the determination of the second approximation for all roots of Equation (1.1), so that it becomes necessary to use such $Z = K' a^{s'}$ as well, for which $s < s' < S_M'$ (the existence of such s' for the cases in question can be proved).

Then the inequality

$$\varepsilon(a) = a^{s'-s} > a^{(q_l - q)/r}$$

will be valid for some roots, and it becomes a matter of interest to establish when this will be the case.

3. a) Suppose K_i is a non-repeated root of the equation $\phi(K) = 0$. Then $\beta_{n-1} - \beta_n$ will be the smallest of the numbers (2.3) and

$$s_M' = \frac{q_l - q}{1} + s$$

Hence

$$\varepsilon(a) = a^{q_l - q} \qquad (l \geqslant 1) \tag{2.5}$$

b) Suppose K_i is a root, repeated r times, of the equation $\phi(K) = 0$ and assume that all $\phi_m(K_i) \neq 0$. Then the relation (2.4) will be valid again for all $u < r$. Indeed, let $\beta_{n-u} = q_{l'} - us$; then, with u increasing, the numbers $q_{l'}$ will not be decreasing and

$$\frac{\beta_{n-u} - \beta_n}{u} = \frac{q_{l'} - us - q_1}{u} = \frac{q_{l'} - q_1}{u} - s > \frac{q - q_1}{r} - s = \frac{q - rs - q_1}{r} = \frac{\beta_{n-r} - \beta_n}{r}$$

Consequently

$$\varepsilon(a) = a^{(q_1 - q)/r} \tag{2.6}$$

c) If K_i is a root, repeated r times, while some $\phi_m(K)$ vanish for $K = K_i$ (speaking generally this happens quite rarely), then with $u < r$ the numbers $q_{l'}$ may be decreasing, while u is increasing, so that there arise two possibilities:

c1) The relationship (2.4) remains valid for all $u < r$, and the equation for the determination of the lowest term Z_M is of degree r. In this case the asymptotic error can be obtained, for each of the r roots considered, which are equal to each other in first approximation, from the formula

$$\varepsilon(a) = a^{(q_l - q)/r} \qquad (l \geqslant 1) \tag{2.7}$$

c2) The relationship (2.4) becomes invalid for some $u < r$. For the determination of the lowest term Z_M from (2.2) we have then the nodal number $U_M' < (\beta_{n-r} - \beta_n)/r$, and for K_i' we obtain an equation of degree $r' < r$. In this case there will be among the roots (repeated r times in first approximation) such ones, for which

$$\varepsilon(a) < a^{(q_l - q)/r}$$

(e.g. such roots for the second approximation of which the lowest term Z_M is being taken).

However, there will necessarily be some (see note at the end of this section) whose asymptotic error will exceed that given in (2.7); i.e.

$$\varepsilon(a) > a^{(q_l - q)/r}$$

and in order to find it we will have to solve Equation (2.2) or, at least, to determine the nodal numbers for the latter.

Let us use an example for illustration. Suppose we have the equation

$$f(a, k) \equiv k^2 - (2\sqrt{a} - a\sqrt{a} - a^3\sqrt{a})\, k + a - a^2 = 0 \tag{2.8}$$

Its two roots are equal to each other in first approximation:

$$k_1 \approx \sqrt{a}, \qquad k_2 \approx \sqrt{a}$$

Since $f(a, Ka^{1/2}) \equiv (K-1)^2 a + (K-1)a^2 + Ka^4$, we have $q = 1$, $q_1 = 2$, $q_2 = 4$.

It is also easily seen that $q_l = q_2 = 4$. Substitute $\sqrt{a} + z$ into Equation (2.8):

$$z^2 + (a^{3/2} + a^{7/2})z + a^4 = 0$$

The nodal numbers of this equation are $-3/2$ and $-5/2$, so that

$$z_1 \approx K_1' a^{5/2}, \qquad z_2 \approx K_2' a^{3/2}$$

Hence we obtain for the roots k_1 and k_2

$$\varepsilon_1(a) = a^2 < a^{(4-1)/2} = a^{3/2}, \qquad \varepsilon_2(a) = a > a^{3/2}$$

respectively. This can be verified by solving the quadratic equation (2.8) directly.

Thus we see that the problem of determining the asymptotic error does not require, in the majority of cases [see (a), (b)], an analysis of Equation (2.2) for determination of z; it is possible to find $\epsilon(a)$ from the formulas (2.5) or (2.6) starting only from the relation (2.1). In the particular case (c) it becomes necessary, for determination of $\epsilon(a)$, to find the order s' of the second term of the series (1.6) starting from (2.2).

4. The following condition should be taken into consideration in the procedure of determining the second approximations for the roots of Equation (1.1).

A limiting equation of degree $r \geqslant 1$, where r is the multiplicity of K_i in the equation $\phi(K) = 0$, is obtained in the cases (a), (b), (c1) for the determination of the lowest term $Z_M = K_i' a^{s'_M}$. We note that $s'_{M'} > s$. The remaining s_k', obtained for Equation (2.2), will not be larger than $s(s_k' \leqslant s)$, which can be easily shown starting from the properties of the table (1.5) for the numbers $\beta_0, \beta_1, \ldots, \beta_n$.

In the case (c2) the lowest term Z_M is being determined from an equation of degree $r' < r$, so that such a Z_M does not permit us to obtain the second approximation for all r of the roots which are equal to each other in first approximation. For $r - r'$ roots we have to take, as seen from the given example, such $Z_j' = K_j' a^{s'_j}$ as well, where s_j' does not represent the largest power exponent. Starting, however, from the properties of the table (1.5) it can be shown that in addition to s_M' there will be such $s_j' > s$, and to them, together with s_M' correspond limiting equations of degrees the sum of which is equal to r, while all remaining $s_k' < s$.

Thus we arrive at the following conclusion: for the determination of $K' a^{s'}$, the second term of the series (1.6), we have to use all $s' > s$. This ascertains the derivation of the second approximation for all roots of Equation (1.1).

The formulated statements can be proved starting from the following considerations:

1) In the table (1.5), constructed for the numbers $\beta_0, \beta_1, \ldots, \beta_n$, all nodal numbers, appearing in the last r columns, appear in the last (lowest) r lines; in the cases (a), (b), (c1) there will be only one such nodal number;

2) The rth line, counting from the bottom, contains the nodal number U_m', which gives the smallest of all permissible (smaller than s) power exponents.

Note. Since U_m' represents the largest number in the rth line, counting from the bottom of the table we have

$$U_m' \geqslant (q - q_l) / r - s$$

For the root corresponding to this number we consequently have

$$\varepsilon(a) = a^{s'_m - s} \geqslant a^{(q_l - q)/r}$$

In the cases (a), (b) and (c1) we have $s_m' = s_M'$, and the asymptotic error

$$\varepsilon(a) = a^{s'_m - s} = a^{s'_M - s}$$

is to be computed from the formulas (2.5), (2.6) and (2.7). In the case (c2) we have $s_m' < s_M'$, and for the roots corresponding to s_m' we obtain

$$\varepsilon\,(a) = a^{s'_m - s} > a^{(q_l - q)/r}$$

because

$$U_m' > (q_l - q)\,/\,r - s$$

3. The fundamental formulas in the second approximation. We shall deal here with the formulas for the mth term of the expansion into a series for the case of a state of stress and strain symmetrical with respect to the generator $\theta = 0$ of a circular cylindrical shell. To this end we set $\Phi = \phi \cos m\theta$. By virtue of (0.5) $\Phi = Ae^{k\xi} \cos m\theta$. Substituting Φ into the relations (0.1) we will have

$$u = AP_u\,(k, m)\, e^{k\xi} \cos m\theta, \qquad v = AQ_v\,(k, m)\, e^{k\xi} \sin m\theta,$$
$$w = AQ_w\,(k, m)\, e^{k\xi} \cos m\theta$$

Analogous formulas are obtained for T_1, T_2, S_1, S_2, and so forth, with

$$P_u = \lambda_1 m^2 k + \lambda_2 \sigma k^3 + a^2 \Big[-\lambda_{14} \frac{(1+\sigma)(2-\sigma)}{1-\sigma} m^2 k^3 + \lambda_{14} \frac{1+\sigma}{1-\sigma} m^4 k + 4\sigma k^3 - \frac{2\sigma}{1-\sigma} m^2 k\Big]$$

$$Q_v = \lambda_1 m^3 - \lambda_2 (2+\sigma) m k^2 + a^2 \Big[\lambda_3 \frac{2(2-\sigma)}{1-\sigma} m k^4 - \lambda_{14} \frac{4 - 3\sigma + \sigma^2}{1-\sigma} m^3 k^2 + \lambda_{14} m^5\Big]$$

$$Q_w = \lambda_4 k^4 - 2\lambda_5 m^2 k^2 + \lambda_6 m^4 + a^2 \Big[4k^4 - \frac{2(2 - 2\sigma + \sigma^2)}{1-\sigma} m^2 k^2 + m^4\Big]$$

$$Q_{T_1} = \frac{2Eh}{r(1-\sigma^2)} \{\lambda_7 (1-\sigma^2) m^2 k^2 + a^2 [-\lambda_3 (2-\sigma) m^2 k^4 + \lambda_{14} (1-\sigma)^2 m^4 k^2 + \lambda_{14} \sigma m^6 + 2\sigma m^2 k^2 - \sigma m^4]\}$$

$$Q_{T_2} = \frac{2Eh}{r(1-\sigma^2)} \{-\lambda_7 (1-\sigma^2) k^4 + a^2 [\lambda_{14} (4-\sigma^2) m^2 k^4 - 4\lambda_{14} m^4 k^2 + \lambda_{16} m^6 - 4(1-\sigma^2) k^4 + 4m^2 k^2 - \lambda_9 m^4]\}$$

$$P_{S_1} = -\frac{2Eh}{r(1-\sigma^2)} \{\lambda_7 (1-\sigma^2) m k^3 + a^2 [-\lambda_3 m k^5 - \lambda_{14} (1-\sigma^2) m^3 k^3 + \lambda_{14} m^5 k + (1-\sigma)(2+3\sigma) m k^3 - m^3 k] + a^4 [\sigma (1-\sigma) m^3 k^3 - 2\sigma m k^5]\}$$

$$P_{S_2} = -\frac{2Eh}{r(1-\sigma^2)} \{-\lambda_7 (1-\sigma^2) m k^3 + a^2 [\lambda_3 (2-\sigma) m k^5 - \lambda_{14} (1-\sigma)^2 m^3 k^3 - \lambda_{14} \sigma m^5 k - 2\sigma (1-\sigma) m k^3 + \sigma m^3 k]\} \tag{3.1}$$

$$Q_{G_1} = -\frac{2Eh}{1-\sigma^2} a^2 \{\lambda_4 k^6 - \lambda_{10} (2+\sigma) m^2 k^4 + \lambda_{11} (1+2\sigma) m^4 k^2 - \lambda_6 \sigma m^6 - \sigma (2+\sigma) m^2 k^2 + \lambda_8 \sigma m^4 + a^2 [4k^6 - 4m^2 k^4 + (1-\sigma^2) m^4 k^2]\}$$

$$Q_{G_2} = -\frac{2Eh}{1-\sigma^2} a^2 \{\lambda_4 \sigma k^6 - \lambda_{10}(1+2\sigma) m^2k^4 + \lambda_{11}(2+\sigma) m^4k^2 - \lambda_6 m^6 - $$
$$- (2+\sigma) m^2k^2 + \lambda_8 m^4 + a^2 [4\sigma k^6 - 2\sigma(1-\sigma) m^2k^4]\}$$

$$P_{N_1} = -\frac{2Eh}{r(1-\sigma^2)} a^2 \{\lambda_4 k^7 - 3\lambda_{10} m^2k^5 + 3\lambda_{13} m^4k^3 - \lambda_6 m^6 k -$$
$$- (2+\sigma) m^2k^3 + \lambda_8 m^4 k + a^2 [4k^7 - 2(2-\sigma) m^2k^5 + (1-\sigma) m^4k^3]\}$$

$$Q_{N_2} = \frac{2Eh}{r(1-\sigma^2)} a^2 \{\lambda_4 mk^6 - 3\lambda_{10} m^3k^4 + 3\lambda_{13} m^5k^2 - \lambda_6 m^7 +$$
$$+ \lambda_{15}(1-\sigma)(2+\sigma) mk^4 - 3m^3k^2 + \lambda_8 m^5 + a^2 [2\sigma mk^6 - \sigma(1-\sigma) m^3k^4]\}$$

In these formulas all $\lambda_i = 1$ for any i. They are used here in the interest of convenience of presentation.

Let us study the question of approximate representation of the quantities (3.1). To this end we shall use the method described in Section 1 with reference to the process of solving Equation (0.6). Three cases are to be considered: $\mu < 1/2$; $\mu = 1/2$; $\mu > 1/2$.

1. Take the case $0 < \mu < 1/2$. Table (1.5), constructed for the equation under study, will have two nodal numbers $U = U_1 = 1/2$ and $U = U_2 = 2\mu - 1/2$.

a) Suppose $U = U_1 = 1/2$. Then

$$k_i \approx (k_i)_1 = K_i a^{-1/2} \qquad (i = 1, 2, 3, 4) \tag{3.2}$$

The limiting equation for determination of K_i is of the form

$$K^8 + (1-\sigma^2) K^4 = 0$$

Dividing by K^4 and returning to $(k)_1$ we obtain the equation

$$(k)_1^4 + (1-\sigma^2) / a^2 = 0 \tag{3.3}$$

from which we determine the four roots of Equation (0.6) in first approximation. Since the roots of this equation are non-repeated, their asymptotic error is determined by the formula (2.5):

$$\varepsilon(a) = a^{1-2\mu} \tag{3.4}$$

By virtue of the relations (0.4), (3.2) and (3.4) the formulas (3.1) can be simplified [1], in consequence of which only such terms remain in them which contain λ_2, λ_4 and λ_7 (with k replaced by $(k)_1$). The asymptotic error for the simplified formulas is expressed by

Equation (3.4).

To improve the accuracy of these formulas we shall find the second approximation for the four roots of Equation (0.6) considered. For this purpose we replace k by $(k)_1 + k'$ in Equation (0.6) and, using the same method, we find for k' its first approximation $(k')_1$ from the limiting equation

$$(k)_1\,(k')_1 - m^2 = 0$$

(the coefficient of k' is simplified on the basis of Equation (3.3) and the equation is divided by $(k)_1{}^6$). This gives the expression

$$k_i \approx (k_i)_2 = (k_i)_1 + (k_i')_1 = K_i a^{-1/2} + K_i' a^{1/2-2\mu}$$

for the root and the formula

$$\varepsilon'(a) = a^{2-4\mu} \tag{3.5}$$

for its asymptotic error. Substituting $(k)_1 + (k')_1$ for k into the formulas (3.1), we omit the terms whose asymptotic error does not exceed (3.5).

Then the terms of the first approximation will be amplified by the terms with λ_2, λ_4, λ_7, linear with respect to $(k')_1$, as well as by terms with λ_1, λ_3, λ_5, λ_{10},λ_{15}, free of $(k')_1$, which we can write down symbolically in the following manner:

$$\{\lambda_2, \lambda_4, \lambda_7;\ \lambda\ [(\lambda_2, \lambda_4, \lambda_7)\ (k')_1, \lambda_1, \lambda_3, \lambda_5, \lambda_{10}, \lambda_{15}]\} \tag{3.6}$$

For example, $P_u = \sigma(k)_1{}^3 + \lambda[3\sigma(k)_1{}^2k')_1 + m^2(k)_1]$. Here we hav $\lambda = 0$ for the first and $\lambda = 1$ for the second approximation.

Since, in general, the theory of thin shells itself involves errors of the order a, the question of retaining in the formulas of analysis such terms, which are of the order a as compared with the main terms, requires additional study. This question will not be discussed here and the terms indicated are everywhere omitted.

Accordingly, we shall omit the terms with λ_3 and λ_{15} in the formulas, which correspond to the scheme (3.6), and in the case $\mu = 0$ we shall disregard in addition all other terms of the second approximation. In this way we obtain (with $\lambda = 0$ when $\mu = 0$) the formulas

$$\{\lambda_2, \lambda_4, \lambda_7;\ \lambda\ [(\lambda_2, \lambda_4, \lambda_7)\ (k')_1, \lambda_1, \lambda_5, \lambda_{10}]\} \tag{3.7}$$

and the error is

$$\varepsilon'(a) = \begin{cases} a^{2-4\mu} & \text{when } \frac{1}{4} < \mu < \frac{1}{2} \\ a & \text{when } 0 \leqslant \mu \leqslant \frac{1}{4} \end{cases}$$

The accuracy of the formulas (3.7) can be improved in an analogous manner. This does not offer any fundamental difficulties, and we shall not discuss here or in the sequel the derivation of the third approximation.

b) Suppose $U = U_2 = 2\mu - 1/2$; then, as in the preceding case, we obtain, if $\mu \neq 0$, for the remaining four roots of Equation (0.6)

$$(1 - \sigma^2)(k)_1^4 + a^2 m^8 = 0, \quad k_i \approx (k_i)_1 = K_i a^{1/2 - 2\mu} \quad (i = 5, 6, 7, 8) \tag{3.8}$$

$$\varepsilon(a) = \begin{cases} a^{2\mu} & \text{when } 0 < \mu < \frac{1}{4} \\ a^{1/2} & \text{when } \mu = \frac{1}{4} \\ a^{1-2\mu} & \text{when } \frac{1}{4} < \mu < \frac{1}{2} \end{cases}$$

1) If $0 < \mu < 1/4$

$$2\,\frac{1-\sigma^2}{a^2}(k)_1^3 (k')_1 - m^6 = 0, \qquad k_i \approx K_i a^{1/2 - 2\mu} + K_i' a^{1/2}$$

$$\varepsilon'(a) = \begin{cases} a^{4\mu} & \text{when } 0 < \mu < \frac{1}{6} \\ a^{2/3} & \text{when } \mu = \frac{1}{6} \\ a^{1-2\mu} & \text{when } \frac{1}{6} < \mu < \frac{1}{4} \end{cases} \tag{3.9}$$

$$\{\lambda_1, \lambda_6, \lambda_7;\ \lambda\ [(\lambda_1, \lambda_6, \lambda_7)(k')_1, \lambda_8, \lambda_{16}]\} \tag{3.10}$$

2) If $1/4 < \mu < 1/2$

$$\frac{1-\sigma^2}{a^2}(k)_1 (k')_1 - m^6 = 0, \qquad k_i \approx K_i a^{1/2 - 2\mu} + K'_i a^{3/2 - 4\mu}$$

$$\varepsilon'(a) = \begin{cases} a^{2\mu} & \text{when } \frac{1}{4} < \mu < \frac{1}{3} \\ a^{2/3} & \text{when } \mu = \frac{1}{3} \\ a^{2-4\mu} & \text{when } \frac{1}{3} < \mu < \frac{1}{2} \end{cases} \tag{3.11}$$

$$\{\lambda_1, \lambda_6, \lambda_7;\ \lambda\ [(\lambda_1, \lambda_6, \lambda_7)(k')_1, \lambda_2, \lambda_5, \lambda_{11}, \lambda_{13}]\} \tag{3.12}$$

3) If $\mu = 1/4$

$$2\,\frac{1-\sigma^2}{a^2}(k)_1^3 (\bar{k}')_1 - 2m^6 (k)_1^2 - m^6 = 0, \quad k_i \approx K_i a^{1/2 - 2\mu} + K_i' a^{1/2}$$

$$\varepsilon'(a) = a$$

$$\{\lambda_1, \lambda_6, \lambda_7;\ \lambda\ [(\lambda_1, \lambda_6, \lambda_7)(k')_1, \lambda_2, \lambda_5, \lambda_8, \lambda_{11}, \lambda_{13}, \lambda_{16}]\}$$

These formulas can be used also in the case of $\mu \neq 1/4$ in the vicinity of the point $\mu = 1/4$ (where for them $\epsilon'(a) \approx a$) instead of the formulas (3.10) and (3.12), which give in this case the larger errors (3.9) and (3.11).

If $\mu = 0$, then the four roots considered are $k_i = 0$, as can be seen from Equation (0.6) and from the limiting equation

$$\frac{1-\sigma^2}{a^2}(k)_1^4 + m^4(m^2-1)^2 = 0 \tag{3.13}$$

In this case

$$\Phi = (B_0 + B_1\xi + B_2\xi^2 + B_3\xi^3)\cos m\theta \qquad (B_i = \text{const})$$

Equation (3.13) can be used, however, also for $\mu \neq 0$ in the vicinity of the point $\mu = 0$. It leads in this case to the small roots of Equation (0.6) with a higher degree of accuracy than Equation (3.9), because $m^8 \approx m^6 \approx m^4$ in the case of small μ.

In the case of small values of μ we have $\epsilon(a) = a^{1-2\mu} \approx a$ for Equation (3.13), therefore we are not looking for higher accuracy of the first approximation formula $\{\lambda_1, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{16}\}$.

2. Consider the case $\mu = 1/2$. We have then $U = 1/2$

$$k_i \approx K_i a^{-1/2}, \qquad [(k)_1^2 - m^2]^4 + \frac{1-\sigma^2}{a^2}(k)_1^4 = 0, \qquad \varepsilon(a) = a$$

Since $\epsilon(a) = a$, it is not necessary to improve the accuracy of $\{\lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_{10}, \lambda_{11}, \lambda_{13}\}$. These formulas can be used also in the vicinity of the point $\mu = 1/2$, since they produce a smaller error than (3.7), (3.12) and (3.14).

3. Let $\mu > 1/2$. Then $U = \mu$

$$k_i \approx K_i a^{-\mu}, \qquad [(k)_1^2 - m^2]^4 = 0$$

The limiting equation gives two quadruple roots. Since all $\phi_m(K_i) \neq 0$ in the relation (2.1), established for Equation (0.6), the asymptotic error is determined in the case under consideration by the formula (2.6), and this leads to

$$\varepsilon(a) = \begin{cases} a^{\mu - 1/2} & \text{when } \frac{1}{2} < \mu < 1 \\ a^{1/2} & \text{when } \mu \geqslant 1 \end{cases}$$

([1] gives here instead the formula $\epsilon(a) = a^{-2}m^{-4} = a^{4\mu-2}$ without upper limit for m). We find there, furthermore, for large values of m

($m > a^{-1/2}$), in first approximation the formulas

$$\{\lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_{10}, \lambda_{11}, \lambda_{13}\} \qquad (3.14)$$

without indication of the domain of their applicability; they are correct for $1/2 < \mu \leqslant 5/6$. For $5/6 < \mu < 5/4$ the formulas become somewhat more involved; they assume the form

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \gamma_7, \lambda_{10}, \lambda_{11}, \lambda_{13}, \lambda_{14}, \lambda_{16}\}$$

For $5/4 \leqslant \mu < \infty$ we obtain

$$\{\lambda_3, \lambda_{14}, \lambda_{16}\}$$

The question of raising the accuracy of these formulas will not be discussed here.

Having Φ_i, u_i, v_i, and so forth, for each root k_i of the characteristic equation, we obtain

$$\Phi = \sum_{i=1}^{8} \Phi_i, \qquad u = \sum_{i=1}^{8} A_i P_{ui}(k_i, m)\, e^{k_i \xi} \cos m\theta \quad \text{etc.}.$$

In an analogous manner we can raise the accuracy of the formulas of analysis of open shells as well by exapnding Φ into a trigonometric series in terms of the variable ξ.

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